Smith Normal Forms and Green Operators for Graded Cartan Matrices of Type A Preprojective Algebras

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Abstract

Let Π_{a-1} be the path-length-graded preprojective algebra of Dynkin type A_{a-1}, with graded Cartan matrix $C_a(q)$. The factorization

$$((1+q^2)I - qA)C_a(q) = I + q^a J$$

gives a fixed sine diagonalization and exact spectral data. The principal result is the complete Smith normal form over $\mathbb{Q}[q]$. If $a = 2r+1$, its nonunit invariant factors are q^2-1 and $r-1$ copies of $q^{2a}-1$; if $a = 2r \geq 4$, they are $(q^2-1)(q^a+1)$ and $r-2$ copies of $q^{2a}-1$; for $a = 2$ the Smith form is (1). Hence every determinantal divisor and Fitting ideal is explicit, and every cyclotomic degeneration of the polynomial cokernel is reduced.

We also determine all complex ranks and real inertias, give explicit singular kernel bases, and derive a six-band generic inverse, cyclotomic residues, algebraic pseudodeterminants, exact real condition numbers, and closed Moore–Penrose Green operators at every singular specialization. Away from $q = \pm 1$, the Green operator is sparse plus one rank-one correction. The path model realizes the underlying preprojective algebra as the graded stable endomorphism algebra of $\bigoplus_{\ell=1}^{a-1} K[\varepsilon]/(\varepsilon^\ell)$ over $K[\varepsilon]/(\varepsilon^a)$, while the coefficient matrices of $C_a(q)$ form the integral Chebyshev basis of the adjacency centralizer.

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1. Introduction

Preprojective algebras of Dynkin quivers are finite-dimensional Frobenius algebras whose graded structure controls a broad range of homological and representation-theoretic invariants. Matrix-valued Hilbert series and graded Cartan data have long been part of this picture. In Dynkin type, the standard Hilbert-series identity involving the adjacency matrix and Nakayama permutation was obtained in characteristic zero by Malkin–Ostrik–Vybornov and proved in arbitrary characteristic by Minamoto [8, 9]. For related Hilbert-series results outside the Dynkin case, see [5]. The present type-A matrix is its explicit path-model specialization. Separately, q-Cartan matrices and their Smith forms have proved useful as computable invariants of graded or derived structures [2].

This paper treats the type-A preprojective algebra at a deliberately explicit level. The path basis gives a simple closed formula for the graded Cartan matrix, but its arithmetic degeneration is less visible. The central result is a complete Smith normal form over $\mathbb{Q}[q]$. It packages all specializations at roots of unity at once: every elementary divisor is cyclotomic and squarefree, the rank defect at a primitive root is read off immediately, and the specialized kernel admits a concrete mirror basis.

The same algebra appears naturally from stable Auslander theory. Put $\Lambda_a = K[\varepsilon]/(\varepsilon^a)$ and $M_\ell = \Lambda_a/(\varepsilon^\ell)$. Write $\underline{\mathrm{Hom}}_{\Lambda_a}$ for module homomorphisms modulo those factoring through the projective module Λ_a, and use the analogous notation for stable endomorphisms. The Auslander algebra of Λ_a has the type-A preprojective algebra as its stable quotient; compare [7]. If $G_a = \bigoplus_{\ell=1}^{a-1} M_\ell$, then, with multiplication in the endomorphism algebra given by categorical composition as below,

$$\underline{\mathrm{End}}_{\Lambda_a}(G_a) \cong \Pi_{a-1}.$$

Thus the ungraded Cartan entry is $C_a(1)_{\ell m} = \dim_K \underline{\mathrm{Hom}}_{\Lambda_a}(M_m, M_\ell)$. This stable realization is also the route by which the matrices below arose.

The standard Dynkin factorization gives more than a determinant calculation. The path adjacency matrix and the mirror involution have a common discrete-sine basis, and the same basis diagonalizes $C_a(q)$ for every value of q. This constant spectral decomposition makes the commutativity of the polynomial family visible and gives one mechanism for the complex rank, the real inertia and spectral conditioning, and the operator residues. To determine polynomial Smith data and kernel bases over cyclotomic coefficient

fields, we use a complementary elementary device. After folding by the mirror involution $\ell \mapsto a - \ell$, the matrix is built from

$$H_m(i,j) = q^{|i-j|}[\min(i,j)]_{q^2}.$$

Its inverse is tridiagonal. The mirror tail becomes a one-row perturbation after multiplication by this inverse, so determinants and coordinate kernels reduce to boundary terms. The Smith form is then recovered from the exact cyclotomic rank stratification. The two mechanisms are deliberately kept separate: the sine transform controls spectral and analytic questions over $\mathbb{C}$, while the mirror reduction controls invariant factors over $\mathbb{Q}[q]$. Their agreement at every cyclotomic specialization provides an internal consistency check, but neither argument depends on finite computation.

Coefficient fields and operator conventions. The path model and the formula for $C_a(q)$ are valid over an arbitrary field K. All Smith-form and cokernel statements are made over the principal ideal domain $\mathbb{Q}[q]$; no Smith form over $\mathbb{Z}[q]$ is asserted. Spectral statements are taken over $\mathbb{C}$, real-inertia and conditioning statements over $\mathbb{R}$, and Moore–Penrose inverses use the standard Hermitian inner product. Throughout, "Green operator" means the ordinary inverse at nonsingular parameters and the Moore–Penrose inverse at singular parameters.

Main theorem. The invariant-factor chain can be stated compactly. If $a = 2r + 1$, then

$$\mathrm{SNF}_{\mathbb{Q}[q]}\, C_a(q) = \mathrm{diag}\big(1^{\,r}, q^2 - 1, (q^{2a} - 1)^{\,r-1}\big),$$

whereas, if $a = 2r \geq 4$, then

$$\mathrm{SNF}_{\mathbb{Q}[q]}\, C_a(q) = \mathrm{diag}\big(1^{\,r}, (q^2 - 1)(q^a + 1), (q^{2a} - 1)^{\,r-2}\big).$$

For $a = 2$ the Smith form is (1). This is proved in Theorem 7.2; the exponent on a diagonal entry denotes its multiplicity.

The paper has two tightly connected layers. The algebraic layer combines the explicit path formula, the Dynkin factorization, and the exact cyclotomic nullity profile to determine the Smith normal form over $\mathbb{Q}[q]$, all determinantal divisors, and all Fitting ideals. The reconstruction step is isolated in Lemma 7.1, so the passage from specialized ranks to the polynomial invariant-factor chain is explicit.

The spectral layer uses the fixed sine basis and the complementary mirror reduction to obtain explicit singular kernels, residues, algebraic pseudodeterminants, Moore–Penrose Green operators, real inertia, and exact spectral condition numbers. The two methods overlap at every cyclotomic specialization, but each proof remains independent of finite computation. The coefficient identity in Corollary 3.3 records the integral Chebyshev centralizer structure needed for the sign and reciprocity symmetries.

2. The type-A path basis

Fix a field K and an integer $a \geq 2$. Let Π_{a-1} be the path algebra of the doubled line

$$1 \underset{d_1}{\overset{u_1}{\leftrightarrows}} 2 \underset{d_2}{\overset{u_2}{\leftrightarrows}} \cdots \underset{d_{a-2}}{\overset{u_{a-2}}{\leftrightarrows}} a-1$$

modulo the preprojective relations

$$d_1 u_1 = 0, \qquad d_\ell u_\ell = u_{\ell-1} d_{\ell-1} \ (2 \leq \ell \leq a-2), \qquad u_{a-2} d_{a-2} = 0.$$

Every arrow has degree one. Path products are read in the categorical order: in $d_\ell u_\ell$, the arrow u_ℓ is traversed first. Write e_ℓ for the primitive idempotent at vertex ℓ, and define the graded Cartan matrix by

$$C_a(q)_{\ell m} = \sum_{p \geq 0} \dim_K (e_\ell \Pi_{a-1,p} e_m) q^p.$$

Set

$$[h]_{q^2} = 1 + q^2 + \cdots + q^{2h-2}, \qquad [0]_{q^2} = 0.$$

Proposition 2.1 (normal forms and graded Cartan entries)**.** *For $1 \leq \ell, m < a$, put*

$$h(\ell, m) = \min(\ell, m, a-\ell, a-m).$$

There is exactly one normal-form path from m to ℓ in each degree

$$|\ell - m|, \ |\ell - m| + 2, \ldots, |\ell - m| + 2(h(\ell, m) - 1).$$

Consequently

$$C_a(q)_{\ell m} = q^{|\ell - m|} [h(\ell, m)]_{q^2}. \tag{1}$$

Moreover

$$\sum_{\ell, m=1}^{a-1} C_a(q)_{\ell m} = \sum_{p=0}^{a-2} (p+1)(a-1-p) q^p,$$

so in particular

$$\dim_K \Pi_{a-1} = \binom{a+1}{3}.$$

Proof. We make the normal-form count rigorous through the stable-Auslander model. Put $\Lambda_a = K[\varepsilon]/(\varepsilon^a)$ and $M_j = \Lambda_a/(\varepsilon^j)$ for $1 \le j < a$. A homomorphism $M_m \to M_\ell$ is determined by

$$f_t(1) = \varepsilon^t, \qquad \max(0, \ell - m) \le t \le \ell - 1.$$

Such a map factors through the projective module $M_a = \Lambda_a$ if and only if $t \ge a - m$. Indeed, this inequality gives the factorization

$$M_m \xrightarrow{1 \mapsto \varepsilon^{a-m}} M_a \xrightarrow{1 \mapsto \varepsilon^{t-a+m}} M_\ell,$$

and every map from M_m to M_a has image divisible by ε^{a-m}. Hence the stable Hom space has basis

$$\underline{f}_t, \qquad \max(0, \ell - m) \le t \le \min(\ell - 1, a - m - 1).$$

Its dimension is $h(\ell, m)$.

Give $\underline{f}_t$ degree $2t + m - \ell$. These degrees are additive under composition: whenever $\underline{f}_t \circ \underline{f}_s$ is nonzero, its exponent is $s + t$ and its degree is the sum of the two displayed degrees. Thus they define a grading on the stable endomorphism algebra. For $1 \le j \le a - 2$, let

$$\iota_j : M_j \longrightarrow M_{j+1}, \qquad \iota_j(1) = \varepsilon, \qquad \pi_j : M_{j+1} \longrightarrow M_j, \qquad \pi_j(1) = 1.$$

Give both maps degree one, and send the quiver arrows u_j and d_j to ι_j and π_j, respectively. Endomorphisms are multiplied in categorical order, so this assignment has the same product convention as the path algebra. The composites satisfy

$$\pi_1\iota_1 = 0, \qquad \pi_j\iota_j = \iota_{j-1}\pi_{j-1} = \varepsilon \cdot \quad (2 \le j \le a - 2).$$

The remaining endpoint composite $\iota_{a-2}\pi_{a-2} = \varepsilon\cdot$ on M_{a-1} factors through the projective module Λ_a: it is the composite

$$M_{a-1} \xrightarrow{1 \mapsto \varepsilon} \Lambda_a \xrightarrow{1 \mapsto 1} M_{a-1}.$$

Hence the arrow assignment defines a graded algebra homomorphism

$$\Pi_{a-1} \longrightarrow \underline{\mathrm{End}}_{\Lambda_a}\left(\bigoplus_{j=1}^{a-1} M_j\right),$$

with no opposite-algebra reversal under the stated conventions.

We now verify directly that no additional path classes survive. A path from m to ℓ of length p has

$$U - D = \ell - m, \qquad U + D = p,$$

where U and D are its numbers of right- and left-pointing arrows. Hence $p = |\ell - m| + 2j$ for a unique integer $j \geq 0$.

Assume first that $m \leq \ell$. For a word in U, D, let

$$\mathrm{inv}_{UD}(w) = \#\{(s,t) : s < t,\ w_s = U,\ w_t = D\}.$$

Whenever an adjacent subword UD is based at an interior vertex, the mesh relation replaces it by DU at the same vertex and lowers $\mathrm{inv}_{UD}(w)$ by one. If the subword is based at the left endpoint, the endpoint relation makes the whole path zero. Thus the rewrite terminates at zero or at the unique valley word

$$D^j U^{\ell - m + j}.$$

In particular, it reaches the left endpoint and gives zero when $j \geq m$. Sorting in the opposite direction uses $\mathrm{inv}_{DU}(w) = \#\{(s,t) : s < t,\ w_s = D,\ w_t = U\}$ and the rewrite $DU \mapsto UD$. It terminates at zero or at the unique peak word

$$U^{\ell - m + j} D^j,$$

and reaches the right endpoint when $j \geq a - \ell$. Hence the component is zero for $j \geq \min(m, a - \ell)$ and otherwise is spanned by the valley word.

For $m \geq \ell$ the mirror argument gives the peak word $U^j D^{m - \ell + j}$ and the valley word $D^{m - \ell + j} U^j$; the endpoint relations kill the component when $j \geq a - m$ or $j \geq \ell$. In both cases the vanishing threshold is therefore

$$j \geq \min(\ell, m, a - \ell, a - m) = h(\ell, m).$$

Thus the degree $|\ell - m| + 2j$ component has dimension at most one for $0 \leq j < h(\ell, m)$ and is zero otherwise.

For $0 \le j < h(\ell, m)$ the one-turn path stays inside the line. Under the stable-Auslander homomorphism it sends 1 to $\varepsilon^{\max(0,\ell-m)+j}$, namely to the nonzero basis map $\underline{f}_t$ with $t = \max(0, \ell - m) + j$. The surviving path classes are therefore nonzero and lie in distinct degrees

$$2t + m - \ell = |\ell - m|, |\ell - m| + 2, \ldots, |\ell - m| + 2(h(\ell, m) - 1).$$

They are independent as well as spanning, proving the normal-form claim and (1). On each graded e_ℓ–e_m component, the constructed homomorphism therefore maps a basis of surviving paths bijectively to the stable-Hom basis. Hence it is an isomorphism of graded algebras. Counting the basis maps of degree p gives $(p + 1)(a - 1 - p)$, which proves the final identity. □

Remark 2.2. The proof also establishes the graded stable-Auslander realization

$$\Pi_{a-1} \cong \underline{\mathrm{End}}_{\Lambda_a}\left(\bigoplus_{j=1}^{a-1} \Lambda_a/(\varepsilon^j)\right),$$

where the underline denotes quotienting by maps through the projective module Λ_a. The explicit arrow assignment in the proof fixes the composition convention and shows that no opposite algebra is required.

3. Dynkin factorization and the meromorphic Green operator

Let A_a be the adjacency matrix of the path on vertices $1, \ldots, a - 1$, let $J_a e_\ell = e_{a-\ell}$, and put

$$K_a(q) = (1 + q^2)I - qA_a, \qquad P_\pm = \frac{I \pm J_a}{2}.$$

The matrix J_a is the Nakayama permutation in type A. The next identity is the type-A specialization of the standard Dynkin preprojective Hilbert-series formula [8, 9]; we include an entrywise proof because it drives the later Green-operator formulas.

Theorem 3.1 (Dynkin factorization and generic inverse)**.** *Over* $\mathbb{Z}[q]$ *one has*

$$K_a(q)C_a(q) = C_a(q)K_a(q) = I + q^a J_a. \tag{2}$$

Consequently, whenever $q^{2a} \neq 1$,

$$C_a(q)^{-1} = K_a(q)\frac{I - q^a J_a}{1 - q^{2a}}, \tag{3}$$

$$= K_a(q)\left(\frac{P_+}{1 + q^a} + \frac{P_-}{1 - q^a}\right). \tag{4}$$

Proof. For $1 \le \ell, m < a$, apply the second-difference operator in the ℓ index to the entry formula (1):

$$(K_a C_a)_{\ell m} = (1+q^2)C_{\ell m} - qC_{\ell-1,m} - qC_{\ell+1,m},$$

where entries outside the path are zero. The three terms cancel unless $\ell = m$ or $\ell = a - m$. The first exceptional point contributes 1, the mirror point contributes q^a, and if the two points coincide the contributions add. Hence $K_a C_a = I + q^a J_a$. All three matrices commute with J_a and are symmetric, so also $C_a K_a = I + q^a J_a$.

Because $J_a^2 = I$,

$$(I + q^a J_a)^{-1} = \frac{I - q^a J_a}{1 - q^{2a}} = \frac{P_+}{1+q^a} + \frac{P_-}{1-q^a}.$$

Multiplying (2) by this inverse proves (3) and (4). □

Theorem 3.2 (constant sine diagonalization)**.** *For $1 \le k < a$, put*

$$\theta_k = \frac{k\pi}{a}, \qquad v_k = \sqrt{\frac{2}{a}}\big(\sin(\ell\theta_k)\big)_{\ell=1}^{a-1}, \qquad \epsilon_k = (-1)^{k+1}.$$

Then $v_1, \dots, v_{a-1}$ is an orthonormal basis, independent of q, and

$$C_a(q)v_k = \lambda_{a,k}(q)v_k, \qquad \lambda_{a,k}(q) = \sum_{j=0}^{a-2} \frac{\sin((j+1)\theta_k)}{\sin\theta_k} q^j = \frac{1+\epsilon_k q^a}{1 - 2q\cos\theta_k + q^2}. \tag{5}$$

The quotient in (5) *is a polynomial, with the apparent poles removable. If $S_a = (v_1 \ \cdots \ v_{a-1})$, then over $\mathbb{C}[q]$*

$$C_a(q) = S_a \ \mathrm{diag}(\lambda_{a,1}(q), \dots, \lambda_{a,a-1}(q)) \, S_a^T.$$

In particular, the matrices $C_a(z)$ commute for all $z \in \mathbb{C}$ and are normal with respect to the standard Hermitian inner product.

Proof. The usual sine transform for the path gives

$$A_a v_k = 2\cos\theta_k \, v_k, \qquad J_a v_k = \epsilon_k v_k.$$

Let U_j be the Chebyshev polynomial of the second kind, so that $U_j(\cos\theta_k) = \sin((j+1)\theta_k)/\sin\theta_k$. The three-term recurrence gives

$$(1 - 2q\cos\theta_k + q^2)\sum_{j=0}^{a-2} U_j(\cos\theta_k) q^j = 1 + U_{a-2}(\cos\theta_k) q^a.$$

Here $U_{a-1}(\cos\theta_k) = 0$ and $U_{a-2}(\cos\theta_k) = \epsilon_k$, proving the polynomial identity in (5). Applying (2) to v_k gives the asserted eigenvalue for generic q, and hence for every q by polynomial continuation. The remaining statements follow from the fixed orthogonal diagonalization. □

Corollary 3.3 (Chebyshev fusion basis and sign symmetry)**.** *Write*

$$C_a(q) = \sum_{p=0}^{a-2} C_{a,p} q^p.$$

Then

$$C_{a,p} = U_p(A_a/2), \qquad 0 \le p \le a-2, \tag{6}$$

where U_p is the Chebyshev polynomial of the second kind. Hence, for $a \ge 3$,

$$C_{a,0} = I, \qquad C_{a,1} = A_a, \qquad C_{a,p+1} = A_a C_{a,p} - C_{a,p-1} \quad (1 \le p \le a-3).$$

The matrices $C_{a,0}, \dots, C_{a,a-2}$ form a $\mathbb{Q}$-basis of the full centralizer of A_a in $M_{a-1}(\mathbb{Q})$:

$$Z_{M_{a-1}(\mathbb{Q})}(A_a) = \mathbb{Q}[A_a] = \operatorname{span}_{\mathbb{Q}}\{C_{a,0}, \dots, C_{a,a-2}\}. \tag{7}$$

They also form the integral centralizer lattice:

$$Z_{M_{a-1}(\mathbb{Z})}(A_a) = \mathbb{Z}[A_a] = \bigoplus_{p=0}^{a-2} \mathbb{Z} C_{a,p}. \tag{8}$$

Put $L = a-2$. For $0 \le p, q \le L$ one has the truncated product law

$$C_{a,p} C_{a,q} = \sum_{\substack{|p-q| \le t \le \min\{p+q,\, 2L-p-q\} \\ t \equiv p+q \pmod 2}} C_{a,t}. \tag{9}$$

Thus the coefficient lattice is an integral based ring, with structure constants in $\{0,1\}$, isomorphic to the level-L type-A or $\widehat{\mathfrak{sl}}_2$ fusion ring [11].

Let

$$D_a = \operatorname{diag}((-1)^1, \dots, (-1)^{a-1}).$$

Then, for every $q \in \mathbb{C}$,

$$C_a(-q) = D_a C_a(q) D_a. \tag{10}$$

Moreover $D_a v_k = -v_{a-k}$ *and*

$$\lambda_{a,k}(-q) = \lambda_{a,a-k}(q). \tag{11}$$

Consequently $C_a(q)$ *and* $C_a(-q)$ *have the same characteristic polynomial, rank, and singular values. Their inverses, whenever defined, and their Moore–Penrose inverses satisfy the same diagonal conjugation law.*

Proof. Comparing coefficients of q^p in (5) gives

$$C_{a,p} = S_a \operatorname{diag}\big(U_p(\cos\theta_1), \dots, U_p(\cos\theta_{a-1})\big) S_a^T = U_p(A_a/2),$$

which proves (6) and the recurrence. The path adjacency matrix has the $a-1$ distinct eigenvalues $2\cos(k\pi/a)$, so its minimal polynomial has degree $a-1$ and its centralizer is $\mathbb{Q}[A_a]$. Since $U_p(A_a/2) = A_a^p$ plus lower-degree terms, the displayed coefficient matrices form a basis, proving (7).

For the integral statement, let e_1 be the first coordinate vector. The Krylov matrix

$$\mathcal{K}_a = (e_1, A_a e_1, \dots, A_a^{a-2} e_1)$$

is unitriangular over $\mathbb{Z}$: the $(j+1)$st entry of $A_a^j e_1$ is one, from the unique monotone walk, and all entries below it vanish. If an integral matrix X commutes with A_a, then $X = \sum_{j=0}^{a-2} \alpha_j A_a^j$ with $\alpha_j \in \mathbb{Q}$ by (7), while $\mathcal{K}_a(\alpha_0, \dots, \alpha_{a-2})^T = X e_1 \in \mathbb{Z}^{a-1}$. Since $\mathcal{K}_a^{-1}$ is integral, every α_j lies in $\mathbb{Z}$. Monicity of $U_p(x/2)$ now gives (8).

Write $F_m = U_m(A_a/2)$. The ordinary Clebsch–Gordan identity

$$U_p(x) U_q(x) = \sum_{j=0}^{\min(p,q)} U_{p+q-2j}(x)$$

holds in $\mathbb{Z}[x]$. Here $F_{L+1} = U_{a-1}(A_a/2) = 0$, and the Chebyshev recurrence gives

$$F_{L+1+s} = -F_{L+1-s} \qquad (1 \le s \le L+1).$$

In the unrestricted product sum, the terms above L therefore cancel their reflections below L, while the possible middle term F_{L+1} vanishes. The surviving indices are exactly those in (9).

The bipartite sign matrix satisfies $D_a A_a D_a = -A_a$, while $U_p(-x) = (-1)^p U_p(x)$. Therefore

$$D_a C_a(q) D_a = \sum_{p=0}^{a-2} (-1)^p C_{a,p} q^p = C_a(-q),$$

which is (10). The identity $D_a v_k = -v_{a-k}$ gives (11). The remaining claims follow from orthogonal conjugation. □

Corollary 3.4 (graded Frobenius reciprocity)**.** *As an identity of Laurent-polynomial matrices,*

$$C_a(q) = q^{a-2} J_a C_a(q^{-1}) = q^{a-2} C_a(q^{-1}) J_a. \tag{12}$$

Equivalently, each sine eigenvalue satisfies

$$\lambda_{a,k}(q) = \epsilon_k q^{a-2} \lambda_{a,k}(q^{-1}). \tag{13}$$

If $C_a(q) = \sum_{p=0}^{a-2} C_{a,p} q^p$*, then*

$$C_{a,p} = J_a C_{a,a-2-p} = C_{a,a-2-p} J_a.$$

In particular, $C_{a,0} = I$ *and* $C_{a,a-2} = J_a$*. Taking determinants gives*

$$\det C_a(q) = (-1)^{\lfloor (a-1)/2 \rfloor} q^{(a-1)(a-2)} \det C_a(q^{-1}). \tag{14}$$

Hence $\det C_a(q)$ *has degree exactly* $(a-1)(a-2)$*, constant coefficient* 1*, and leading coefficient* $(-1)^{\lfloor (a-1)/2 \rfloor}$*. Thus the entry formula, the mirror involution, and the top-degree Frobenius pairing encode the same Nakayama symmetry.*

Proof. From (5),

$$\epsilon_k q^{a-2} \lambda_{a,k}(q^{-1}) = \epsilon_k q^{a-2} \frac{1 + \epsilon_k q^{-a}}{1 - 2q^{-1} \cos\theta_k + q^{-2}} = \frac{1 + \epsilon_k q^a}{1 - 2q \cos\theta_k + q^2} = \lambda_{a,k}(q).$$

Because $J_a v_k = \epsilon_k v_k$, the fixed sine diagonalization converts this scalar reciprocity into (12). Comparing coefficients gives the coefficient identities. Finally, J_a is the reversal permutation on $a-1$ letters, so $\det J_a = (-1)^{(a-1)(a-2)/2} = (-1)^{\lfloor (a-1)/2 \rfloor}$. Taking determinants in (12) proves (14); its degree and endpoint coefficients follow from $\det C_a(0) = 1$. □

Corollary 3.5 (spectral kernels)**.** *For* $z \in \mathbb{C}$*, define*

$$I_a(z) = \left\{ k \in \{1, \dots, a-1\} : z^a = (-1)^k, \quad z \neq e^{i\theta_k}, e^{-i\theta_k} \right\}.$$

Then

$$\ker C_a(z) = \operatorname{span}\{v_k : k \in I_a(z)\}.$$

Equivalently, the roots of $\lambda_{a,k}$ *are precisely the roots of* $q^a = (-1)^k$ *other than* $e^{\pm i\theta_k}$*, and every such root is simple.*

Proof. The denominator in (5) factors as $(q - e^{i\theta_k})(q - e^{-i\theta_k})$. These two simple roots also occur in $1 + \epsilon_k q^a$ and are exactly the factors cancelled in the polynomial quotient. All remaining numerator roots are simple. The fixed orthonormal diagonalization then gives the kernel formula. □

Corollary 3.6 (sparse generic Green kernel)**.** *Let δ_{ij} be the Kronecker symbol, interpreted as zero when an index lies outside $\{1, \dots, a-1\}$. If $q^{2a} \neq 1$, then*

$$(C_a(q)^{-1})_{\ell m} = \frac{1}{1-q^{2a}}\Big[(1+q^2)\delta_{\ell m} - q(\delta_{\ell,m-1} + \delta_{\ell,m+1}) \\ - q^a(1+q^2)\delta_{\ell,a-m} + q^{a+1}(\delta_{\ell,a-m-1} + \delta_{\ell,a-m+1})\Big]. \tag{15}$$

Thus the generic inverse is supported on the main diagonal and its first neighbors together with the mirror anti-diagonal and its first neighbors.

Proof. Expand (3) entrywise. The matrix $K_a(q)$ has only its main and first off-diagonals, while $(K_a(q)J_a)_{\ell m} = K_a(q)_{\ell,a-m}$. This gives (15). □

Corollary 3.7 (determinant from the Dynkin factorization)**.** *One has*

$$\det K_a(q) = [a]_{q^2} = \frac{1-q^{2a}}{1-q^2}.$$

Taking determinants in (2) *gives the formulas of Theorem 5.2 below.*

Proof. The leading-principal-minor recurrence for the tridiagonal matrix K_a is $D_j = (1+q^2)D_{j-1} - q^2 D_{j-2}$ with $D_0 = 1$ and $D_1 = 1 + q^2$; hence $D_j = [j+1]_{q^2}$. The eigenvalue multiplicities of J_a are the dimensions of the mirror-symmetric and mirror-antisymmetric subspaces. Therefore $\det(I + q^a J_a)$ is the corresponding product of powers of $1 + q^a$ and $1 - q^a$, and division by $\det K_a$ yields Theorem 5.2. □

4. A half-line q-Green matrix

For $m \geq 1$, define

$$H_m(i,j) = q^{|i-j|}[\min(i,j)]_{q^2} \qquad (1 \leq i, j \leq m),$$

and let T_m be the tridiagonal matrix

$$T_m = \begin{pmatrix} 1+q^2 & -q & & & 0 \\ -q & 1+q^2 & -q & & \\ & \ddots & \ddots & \ddots & \\ & & -q & 1+q^2 & -q \\ 0 & & & -q & 1 \end{pmatrix}. \tag{16}$$

For $m = 1$ we set $T_1 = (1)$; the displayed tridiagonal pattern is for $m \geq 2$. For integers $a, m \geq 1$ with $a \geq 2m$, also put

$$R_m^{(a)}(i,j) = q^{a-i-j}[\min(i,j)]_{q^2}.$$

The inequality $a \geq 2m$ is exactly the range used below and ensures that $R_m^{(a)}$ has entries in $\mathbb{Z}[q]$.

Lemma 4.1 (finite-difference inverse)**.** *For every $m \geq 1$,*

$$T_m H_m = H_m T_m = I_m.$$

In particular H_m is unimodular over $\mathbb{Z}[q]$ and $\det H_m = 1$. Moreover $T_m R_m^{(a)}$ has only diagonal entries and a final row:

$$\begin{aligned} (T_m R_m^{(a)})_{ii} &= q^a && (1 \leq i < m), \\ (T_m R_m^{(a)})_{mm} &= q^{a-2m}, && \\ (T_m R_m^{(a)})_{mj} &= -q^{a-m-j}(q^{2j}-1) && (1 \leq j < m), \end{aligned}$$

and every other entry is zero.

Proof. For an interior row of T_m, the three-term combination

$$-qH_m(i-1,j) + (1+q^2)H_m(i,j) - qH_m(i+1,j)$$

vanishes away from $i = j$ and equals one at $i = j$. The first and final rows satisfy the corresponding one-sided identities. This proves $T_m H_m = I$; symmetry gives the other product. Applying the same finite difference to $R_m^{(a)}$ cancels every interior entry. The surviving diagonal and final boundary terms are the displayed expressions. □

5. Mirror reduction and determinant

Let $J(e_\ell) = e_{a-\ell}$. Since $C_a(q)$ commutes with J, it decomposes into mirror-symmetric and mirror-antisymmetric parts over any coefficient ring in which 2 is invertible.

Lemma 5.1 (explicit mirror row reductions)**.** *Suppose first that $a = 2r+1$, and use the mirror bases $e_j \pm e_{a-j}$, $1 \le j \le r$. After premultiplication by the unimodular matrix T_r, the two mirror blocks become $B_\pm = I_r \pm T_r R_r^{(a)}$, where*

$$(B_\pm x)_i = (1 \pm q^a)x_i \qquad (1 \le i < r), \tag{17}$$

$$(B_\pm x)_r = \mp \sum_{j=1}^{r-1} q^{r+1-j}(q^{2j}-1)x_j + (1 \pm q)x_r. \tag{18}$$

If $a = 2r$ with $r \ge 2$, the antisymmetric block, in the basis $e_j - e_{a-j}$, $1 \le j < r$, is row-equivalent to

$$(B_- x)_i = (1 - q^a)x_i \qquad (1 \le i < r-1), \tag{19}$$

$$(B_- x)_{r-1} = \sum_{j=1}^{r-2} q^{r+1-j}(q^{2j}-1)x_j + (1-q^2)x_{r-1}. \tag{20}$$

For the symmetric block write a vector as $\sum_{j=1}^{r-1} x_j(e_j + e_{a-j}) + z e_r$. Its kernel equations are row-equivalent to

$$(1+q^a)x_i = 0 \qquad (1 \le i < r-1), \tag{21}$$

$$-\sum_{j=1}^{r-2} q^{r+1-j}(q^{2j}-1)x_j + (1+q^2)x_{r-1} + qz = 0, \tag{22}$$

$$2\sum_{j=1}^{r-1} q^{r-j}[j]_{q^2} x_j + [r]_{q^2} z = 0. \tag{23}$$

Proof. The odd and even antisymmetric formulas are the corresponding specializations of Lemma 4.1. For the even symmetric block, its paired corner is $H_{r-1} + R_{r-1}^{(a)}$ and its central column is $c_i = q^{r-i}[i]_{q^2}$. Direct finite differencing gives $T_{r-1}c = qe_{r-1}$, which yields (21) and (22). The central row is $2c^T x + [r]_{q^2} z = 0$, giving (23). □

Theorem 5.2 (determinant)**.** *If $a = 2r + 1$ is odd, then*

$$\det C_a(q) = (1 - q^2)(1 - q^{2a})^{r-1}. \tag{24}$$

If $a = 2r$ is even and $r \geq 2$, then

$$\det C_a(q) = (1 - q^2)(1 - q^a)^{r-2}(1 + q^a)^{r-1}. \tag{25}$$

For $a = 2$, $C_2(q) = (1)$.

Proof. Suppose first that $a = 2r + 1$. Order the vertices as $1, \ldots, r, a - 1, \ldots, a - r$. Then

$$C_a(q) = \begin{pmatrix} H_r & R_r^{(a)} \\ R_r^{(a)} & H_r \end{pmatrix}.$$

Mirror sum and difference give the two blocks $H_r \pm R_r^{(a)}$. By Lemma 5.1, their determinants are the products of the displayed boundary pivots. Hence

$$\det(H_r + R_r^{(a)}) = (1 + q)(1 + q^a)^{r-1},$$

$$\det(H_r - R_r^{(a)}) = (1 - q)(1 - q^a)^{r-1}.$$

Their product is (24).

Now let $a = 2r$. The antisymmetric mirror block is $H_{r-1} - R_{r-1}^{(a)}$, whose determinant is

$$(1 - q^2)(1 - q^a)^{r-2}.$$

For the symmetric block, premultiply its first $r - 1$ rows by T_{r-1}. This preserves the determinant because T_{r-1} is unimodular. The row form in Lemma 5.1 is block lower triangular:

$$\begin{pmatrix} (1 + q^a)I_{r-2} & 0 & 0 \\ * & 1 + q^2 & q \\ * & 2q[r-1]_{q^2} & [r]_{q^2} \end{pmatrix}.$$

Consequently its determinant is

$$\begin{aligned}(1 + q^a)^{r-2}\big((1 + q^2)[r]_{q^2} - 2q^2[r-1]_{q^2}\big) &= (1 + q^a)^{r-2}(1 + q^{2r}) \\ &= (1 + q^a)^{r-1}.\end{aligned}$$

Multiplication by the antisymmetric determinant proves (25). □

6. Cyclotomic ranks and kernels

Let $\zeta \in \mathbb{C}$ be a root of unity. The fixed sine basis gives the rank by counting the uncancelled scalar eigenvalue zeros; the mirror reduction will then supply coordinate kernel bases.

Theorem 6.1 (rank at roots of unity)**.** *Let $a = 2r + 1$ be odd. Then*

$$\operatorname{rank} C_a(\zeta) = \begin{cases} r, & \zeta^2 = 1, \\ r+1, & \zeta^2 \neq 1 \text{ and } \zeta^{2a} = 1, \\ 2r, & \zeta^{2a} \neq 1. \end{cases}$$

Let $a = 2r$ be even. Then

$$\operatorname{rank} C_a(\zeta) = \begin{cases} r, & \zeta^2 = 1 \text{ or } \zeta^a = -1, \\ r+1, & \zeta^2 \neq 1 \text{ and } \zeta^a = 1, \\ 2r-1, & \text{otherwise.} \end{cases}$$

Proof. By Corollary 3.5, singular modes are the indices $1 \leq k < a$ satisfying $\zeta^a = (-1)^k$, except for the unique mode whose angle equals the argument of ζ up to sign. That exceptional mode is present precisely when $\zeta \neq \pm 1$ and $\zeta^{2a} = 1$.

If $a = 2r + 1$, the index set $\{1, \ldots, 2r\}$ contains r indices of each parity. Thus $\zeta = \pm 1$ gives nullity r, a nonreal root of $\zeta^{2a} = 1$ gives nullity $r - 1$ after the one cancellation, and all other parameters give nullity zero. Subtracting from the matrix size $2r$ gives the odd table.

If $a = 2r$, there are $r - 1$ even indices and r odd indices in $\{1, \ldots, 2r - 1\}$. At $\zeta = \pm 1$ the relevant even set has size $r - 1$. If $\zeta^a = 1$ and $\zeta^2 \neq 1$, one even mode is cancelled, so the nullity is $r - 2$. If $\zeta^a = -1$, one of the r odd modes is cancelled, so the nullity is $r - 1$. The complementary ranks give the even table. □

Corollary 6.2 (complete complex rank stratification)**.** *For an arbitrary $z \in \mathbb{C}$, the rank is given by the table in Theorem 6.1 whenever z satisfies one of its displayed root-of-unity conditions, and $C_a(z)$ has full rank otherwise. Equivalently, every singular complex specialization is already one of the cyclotomic specializations classified there.*

Proof. The determinant formulas in Theorem 5.2 show that every complex singular parameter is a root of $q^{2a} - 1$ (with $q = \pm 1$ included). The rank is then supplied by Theorem 6.1. □

Corollary 6.3 (cyclotomic Green residues)**.** *Let $\zeta^{2a} = 1$ and put $\sigma = -\zeta^a \in \{\pm 1\}$. The generic inverse has at most a simple pole at $q = \zeta$ on the mirror sector P_σ. The pole is genuine exactly when $I_a(\zeta) \neq \varnothing$, equivalently when $C_a(\zeta)$ is singular; otherwise the apparent singularity is removable. If $d_k(q) = 1 - 2q\cos\theta_k + q^2$, then*

$$\operatorname*{Res}_{q=\zeta} C_a(q)^{-1} = -\frac{\zeta}{a} \sum_{k \in I_a(\zeta)} d_k(\zeta) v_k v_k^T \tag{26}$$

$$= -\frac{\zeta}{a} K_a(\zeta) P_\sigma. \tag{27}$$

Its image is exactly $\ker C_a(\zeta)$*. In particular, every cyclotomic rank defect is a transverse, reduced degeneration in the operator-valued resolvent.*

Proof. For $k \in I_a(\zeta)$, the numerator $1+\epsilon_k q^a$ has derivative $-a\zeta^{-1}$ at ζ. Hence

$$\lambda'_{a,k}(\zeta) = -\frac{a}{\zeta d_k(\zeta)}, \qquad \operatorname*{Res}_{q=\zeta} \lambda_{a,k}(q)^{-1} = -\frac{\zeta}{a} d_k(\zeta).$$

If $I_a(\zeta) = \varnothing$, the spectral sum is zero and every scalar eigenvalue inverse is holomorphic at ζ, so the apparent singularity is removable. Otherwise the nonzero residue gives a genuine simple pole. The constant sine diagonalization proves (26); its image is visibly the span of the corresponding v_k, which is $\ker C_a(\zeta)$ by Corollary 3.5. Finally, $K_a(\zeta) v_k = d_k(\zeta) v_k$, and the vectors in $I_a(\zeta)$ are exactly the σ-mirror sine modes. This gives (27). □

Corollary 6.4 (cyclotomic Moore–Penrose Green operator)**.** *Let $\zeta^{2a} = 1$ and put $\sigma = -\zeta^a \in \{\pm 1\}$. If $\zeta \neq \pm 1$, let $\kappa = \kappa(\zeta)$ be the unique index $1 \leq \kappa < a$ for which $\zeta = e^{i\theta_\kappa}$ or $\zeta = e^{-i\theta_\kappa}$, and let $E_\zeta = v_\kappa v_\kappa^T$ be the orthogonal projection onto* $\ker K_a(\zeta)$*. Then*

$$C_a(\zeta)^+ = \frac{1}{2} K_a(\zeta) P_{-\sigma} + \frac{1-\zeta^2}{a} E_\zeta. \tag{28}$$

At $\zeta = \pm 1$ the corresponding formula is

$$C_a(\zeta)^+ = \frac{1}{2} K_a(\zeta) P_{-\sigma}. \tag{29}$$

Thus every cyclotomic Moore–Penrose inverse is the sum of an operator supported on three diagonal and three mirror anti-diagonal bands and, at a nonreal root, one rank-one term. If $C_a(\zeta)$ is nonsingular, the same formula is its ordinary inverse.

Proof. Use the fixed real orthogonal sine basis. On a mode with $\epsilon_k = -\sigma = \zeta^a$, the factorization (2) gives

$$d_k(\zeta)\lambda_{a,k}(\zeta) = 2,$$

so the scalar Moore–Penrose inverse is $d_k(\zeta)/2$. These modes give $\frac{1}{2}K_a(\zeta)P_{-\sigma}$.

On the sector $\epsilon_k = \sigma$, the same product is zero. If $\zeta \neq \pm 1$, the matrix $K_a(\zeta)$ has the unique zero mode v_κ. Every other mode in that sector has $\lambda_{a,k}(\zeta) = 0$. At the cancelled mode, l'Hopital's rule in (5) gives

$$\lambda_{a,\kappa}(\zeta) = \frac{-a\zeta^{-1}}{\zeta - \zeta^{-1}} = \frac{a}{1-\zeta^2},$$

whose scalar inverse is $(1-\zeta^2)/a$. This proves (28). At $\zeta = \pm 1$, no sine angle equals 0 or π, so $K_a(\zeta)$ has no cancelled mode and the entire P_σ sector is killed by $C_a(\zeta)$, proving (29). Since the sine diagonalization is unitary, these scalar prescriptions are exactly the Moore–Penrose inverse. □

Corollary 6.5 (cyclotomic pseudodeterminants)**.** *For a complex normal matrix M, let* $\operatorname{pdet} M$ *denote the product of its nonzero eigenvalues, counted with algebraic multiplicity. Let $\zeta^{2a} = 1$. If $a = 2r+1$, then*

$$\operatorname{pdet} C_a(\zeta) = \begin{cases} 2^r, & \zeta = \pm 1, \\ \dfrac{a2^{r-1}}{1-\zeta^{a+1}}, & \zeta \neq \pm 1. \end{cases} \tag{30}$$

If $a = 2r \geq 4$, then

$$\operatorname{pdet} C_a(\zeta) = \begin{cases} 2^{r-1}, & \zeta = \pm 1, \\ \dfrac{a2^{r-1}}{1-\zeta^2}, & \zeta \neq \pm 1,\ \zeta^a = 1, \\ a2^{r-2}, & \zeta^a = -1. \end{cases} \tag{31}$$

For $a = 2$, one has $\operatorname{pdet} C_2(\zeta) = 1$. *Since every $C_a(\zeta)$ is normal, the product of its nonzero singular values is the absolute value of the corresponding expression above.*

Proof. Put $d_k(q) = 1 - 2q\cos\theta_k + q^2$ and $\epsilon_k = (-1)^{k+1}$. The proof of Corollary 6.4 identifies the nonzero eigenvalues at a nonreal root as

$$\frac{2}{d_k(\zeta)} \quad (\epsilon_k = \zeta^a), \qquad \frac{a}{1-\zeta^2}$$

for the unique cancelled sine mode. At $\zeta = \pm 1$ only the first family is present. The parity-sector products of the tridiagonal eigenvalues are the following polynomial identities (the displayed quotients have removable endpoint values):

$$\prod_{\epsilon_k=1} d_k(q) = \begin{cases} \dfrac{1+q^a}{1+q}, & a \text{ odd}, \\ 1+q^a, & a \text{ even}, \end{cases}$$

$$\prod_{\epsilon_k=-1} d_k(q) = \begin{cases} \dfrac{1-q^a}{1-q}, & a \text{ odd}, \\ \dfrac{1-q^a}{1-q^2}, & a \text{ even}. \end{cases}$$

Indeed, $d_k(q) = (q - e^{i\theta_k})(q - e^{-i\theta_k})$, and separating odd from even k partitions the roots of $1 + q^a$ and $1 - q^a$, with the displayed real factors removed when necessary. Multiplying the nonzero eigenvalues and substituting $\zeta^a = \pm 1$ gives (30) and (31). The case $a = 2$ is immediate. □

For explicit kernels, write

$$[n]_q = 1 + q + \cdots + q^{n-1}, \qquad A_{2j}(q) = \frac{q^{2j} - 1}{q + 1}.$$

All coefficients in the next statement are evaluated at $q = \zeta$.

Proposition 6.6 (explicit specialized kernels)**.** *If $a = 2r + 1$, then:*

- *at $\zeta = 1$, a basis is $e_j - e_{a-j}$, $1 \le j \le r$;*
- *at $\zeta = -1$, a basis is $e_j + e_{a-j}$, $1 \le j \le r$;*
- *if $\zeta \neq \pm 1$ and $\zeta^a = 1$, a basis is*

$$e_j - e_{a-j} + q^{r+1-j}[2j]_q(e_r - e_{r+1}), \qquad 1 \le j < r;$$

- *if $\zeta \neq \pm 1$ and $\zeta^a = -1$, a basis is*

$$e_j + e_{a-j} + q^{r+1-j} A_{2j}(q)(e_r + e_{r+1}), \qquad 1 \le j < r.$$

In every other odd case the kernel is zero.

If $a = 2r$, then:

- *if* $\zeta^2 = 1$, *a basis is* $e_j - e_{a-j}$, $1 \le j < r$;
- *if* $\zeta^2 \neq 1$ *and* $\zeta^a = 1$, *a basis is*

$$e_j - e_{a-j} + q^{r+1-j}[j]_{q^2}(e_{r-1} - e_{r+1}), \qquad 1 \le j < r-1;$$

- *if* $\zeta^a = -1$, *a basis is*

$$e_j + e_{a-j} + q^{r-j}(q^{2j} - 1)e_r, \qquad 1 \le j < r.$$

In every other even case the kernel is zero.

Proof. At $q^a = 1$ in odd length, the antisymmetric boundary equation is

$$\sum_{j=1}^{r-1} q^{r+1-j}(q^{2j} - 1)x_j + (1-q)x_r = 0.$$

Choosing one $x_j = 1$ at a time gives $x_r = q^{r+1-j}[2j]_q$. At $q^a = -1$, the symmetric boundary equation gives $x_r = q^{r+1-j}A_{2j}(q)$. When $q = 1$ or $q = -1$, the corresponding whole mirror block vanishes, giving the first two odd bases.

For even a, (20) gives the stated antisymmetric correction because $(q^{2j} - 1)/(q^2 - 1) = [j]_{q^2}$. At $q^a = -1$, set one symmetric coordinate $x_j = 1$ and put $z = q^{r-j}(q^{2j} - 1)$. The boundary equation vanishes, and the central equation vanishes by the polynomial identity

$$2q^{r-j}[j]_{q^2} + [r]_{q^2}q^{r-j}(q^{2j} - 1) = q^{r-j}[j]_{q^2}(q^{2r} + 1).$$

The vectors in every displayed family are independent by their first nonzero mirror coordinates. Their numbers agree with the nullities in Theorem 6.1, so they are bases. □

7. Smith form and cyclotomic cokernel

We now regard $C_a(q)$ as a matrix over the PID $\mathbb{Q}[q]$.

Lemma 7.1 (local Smith reconstruction)**.** *Let* $R = \mathbb{Q}[q]$, *and let* $M \in M_n(R)$ *have nonzero determinant. For an irreducible polynomial* $\phi \in R$, *let* $\nu_\phi(M)$

be the nullity of the reduction of M over $R/(\phi)$. If $d_1 \mid \cdots \mid d_n$ are the Smith invariant factors and $\alpha_i = v_\phi(d_i)$, then

$$\nu_\phi(M) = \#\{i : \alpha_i > 0\}, \qquad v_\phi(\det M) = \sum_i \alpha_i.$$

In particular, if these two integers are equal, every nonzero local exponent α_i is one. Suppose this equality holds for every irreducible ϕ, and let $e_1 \mid \cdots \mid e_t$ be the nonunit invariant factors, chosen monic. Then

$$e_j = \prod_{\phi:\, \nu_\phi(M) \geq t-j+1} \phi, \qquad 1 \leq j \leq t. \tag{32}$$

In particular, $t = \max_\phi \nu_\phi(M)$.

Proof. Reduce a Smith decomposition of M modulo ϕ. A diagonal entry becomes zero exactly when its ϕ-adic valuation is positive, proving the first formula; the second follows by taking determinants. Equality forces every positive integer α_i to equal one. Thus an irreducible ϕ occurs in exactly the last $\nu_\phi(M)$ nonunit invariant factors. Multiplying the irreducibles that occur in each position gives (32). The first nonunit factor has an irreducible divisor that occurs in all t positions, while no irreducible can occur in more than t positions, so $t = \max_\phi \nu_\phi(M)$. □

Theorem 7.2 (Smith normal form)**.** *If $a = 2r + 1$, the invariant factors of $C_a(q)$ are*

$$\underbrace{1, \ldots, 1}_{r}, \quad q^2 - 1, \quad \underbrace{q^{2a} - 1, \ldots, q^{2a} - 1}_{r-1}.$$

If $a = 2r$ with $r \geq 2$, they are

$$\underbrace{1, \ldots, 1}_{r}, \quad (q^2 - 1)(q^a + 1), \quad \underbrace{q^{2a} - 1, \ldots, q^{2a} - 1}_{r-2}.$$

For $a = 2$, the Smith form is (1)*.*

Proof. By Theorem 5.2, every irreducible divisor of $\det C_a(q)$ is cyclotomic. Fix $\Phi_N(q)$ and a primitive root ζ. Extension of scalars from $\mathbb{Q}[q]/(\Phi_N)$ to $\mathbb{C}$ preserves rank, so the nullity modulo Φ_N is the nullity of $C_a(\zeta)$ in Theorem 6.1. In every case this nullity equals the Φ_N-adic valuation of the determinant. By Lemma 7.1, every nonzero local elementary divisor has exponent one.

Suppose $a = 2r + 1$. Writing $t_N = \nu_{\Phi_N}(C_a)$, the determinant and rank tables give

$$t_1 = t_2 = r, \qquad t_N = r - 1 \quad (N \mid 2a,\ N \notin \{1, 2\}),$$

and all other t_N vanish. Thus $t = r$, and (32) gives

$$e_1 = \Phi_1\Phi_2 = q^2 - 1, \qquad e_2 = \cdots = e_r = \prod_{N \mid 2a} \Phi_N = q^{2a} - 1.$$

Now suppose $a = 2r$. The corresponding profile is

$$t_N = \begin{cases} r - 1, & N = 1, 2, \text{ or } N \mid 2a \text{ and } N \nmid a, \\ r - 2, & N \mid a,\ N \notin \{1, 2\}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $t = r - 1$. Formula (32) gives

$$e_1 = \Phi_1\Phi_2 \prod_{\substack{N \mid 2a \\ N \nmid a}} \Phi_N = (q^2 - 1)(q^a + 1), \qquad e_2 = \cdots = e_{r-1} = \prod_{N \mid 2a} \Phi_N = q^{2a} - 1.$$

The case $a = 2$ is immediate. □

Example 7.3 (the first two nontrivial parity cases). For $a = 4$ and $a = 5$, the entry formula gives

$$C_4(q) = \begin{pmatrix} 1 & q & q^2 \\ q & 1 + q^2 & q \\ q^2 & q & 1 \end{pmatrix}, \qquad C_5(q) = \begin{pmatrix} 1 & q & q^2 & q^3 \\ q & 1 + q^2 & q + q^3 & q^2 \\ q^2 & q + q^3 & 1 + q^2 & q \\ q^3 & q^2 & q & 1 \end{pmatrix}.$$

Their Smith forms over $\mathbb{Q}[q]$ are

$$\mathrm{SNF}(C_4) = \mathrm{diag}\big(1, 1, (q^2 - 1)(q^4 + 1)\big),$$
$$\mathrm{SNF}(C_5) = \mathrm{diag}\big(1, 1, q^2 - 1, q^{10} - 1\big).$$

Thus $a = 4$ already displays the extra factor $q^a + 1$ in the even case, whereas $a = 5$ displays the odd chain $q^2 - 1 \mid q^{2a} - 1$.

Remark 7.4. As fixed in the coefficient-field conventions of the introduction, the Smith form is asserted over $\mathbb{Q}[q]$, a PID. A Smith problem over $\mathbb{Z}[q]$ is genuinely different because $\mathbb{Z}[q]$ is not a PID; no integral Smith form over that ring is claimed here.

Corollary 7.5 (determinantal divisors)**.** *Let $\mathfrak{d}_k(C_a)$ denote the monic greatest common divisor of all $k \times k$ minors of $C_a(q)$, with $\mathfrak{d}_0 = 1$. If $a = 2r+1$, then*

$$\mathfrak{d}_k(C_a) = 1 \quad (0 \le k \le r),$$

and for $1 \le j \le r$,

$$\mathfrak{d}_{r+j}(C_a) = (q^2-1)(q^{2a}-1)^{j-1}.$$

If $a = 2r$, then

$$\mathfrak{d}_k(C_a) = 1 \quad (0 \le k \le r),$$

and for $1 \le j \le r-1$,

$$\mathfrak{d}_{r+j}(C_a) = (q^2-1)(q^a+1)(q^{2a}-1)^{j-1}.$$

Thus every Fitting ideal of $\operatorname{coker} C_a(q)$ *is generated by an explicit product of cyclotomic polynomials.*

Proof. Over the PID $\mathbb{Q}[q]$, the kth determinantal divisor is the product of the first k invariant factors. The formulas follow immediately from Theorem 7.2. □

Corollary 7.6 (reduced cyclotomic degenerations)**.** *Let $R = \mathbb{Q}[q]$, let Φ_N be a cyclotomic polynomial, and let $\nu_a(N)$ be the nullity of $C_a(q)$ after reduction modulo Φ_N as in Theorem 6.1. Then*

$$(\operatorname{coker} C_a)_{(\Phi_N)} \cong \big(R_{(\Phi_N)}/(\Phi_N)\big)^{\nu_a(N)}.$$

Equivalently, the order of vanishing of $\det C_a(q)$ *along $\Phi_N = 0$ is exactly the specialized nullity, and no local elementary divisor contains Φ_N^2.*

Proof. The proof of Theorem 7.2 shows that every nonzero local elementary divisor has Φ_N-adic exponent one and that their number is precisely $\nu_a(N)$. Localizing the Smith decomposition gives the result. □

Corollary 7.7 (cyclotomic cokernel)**.** *The finite-length* $\mathbb{Q}[q]$*-module* $\operatorname{coker} C_a(q)$ *is semisimple; equivalently, it is a direct sum of simple cyclotomic field modules. If* $a = 2r + 1$*, the multiplicity of* $\mathbb{Q}[q]/(\Phi_N)$ *is*

$$m_a(N) = \begin{cases} r, & N = 1 \text{ or } 2, \\ r - 1, & N \mid 2a,\ N \notin \{1, 2\}, \\ 0, & \text{otherwise.} \end{cases}$$

If $a = 2r$*, the multiplicity is*

$$m_a(N) = \begin{cases} r - 1, & N = 1 \text{ or } 2, \\ r - 1, & N \mid 2a \text{ and } N \nmid a, \\ r - 2, & N \mid a,\ N \notin \{1, 2\}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Every Smith factor in Theorem 7.2 is squarefree over $\mathbb{Q}$. Factor $q^n - 1 = \prod_{N|n} \Phi_N(q)$ and apply the Chinese remainder theorem. □

8. Real inertia

Theorem 8.1 (real inertia)**.** *Let* $d = a - 1$ *and* $r = \lfloor a/2 \rfloor$*. We write* $\operatorname{inert} M = (n_+(M), n_-(M), n_0(M))$*. For real* q*,*

$$\operatorname{inert} C_a(q) = \begin{cases} (d, 0, 0), & |q| < 1, \\ (r, 0, d - r), & q = \pm 1, \\ (r, d - r, 0), & |q| > 1. \end{cases}$$

Proof. For real q, the denominator in (5) is

$$1 - 2q \cos \theta_k + q^2 = (q - \cos \theta_k)^2 + \sin^2 \theta_k > 0.$$

Thus the sign of the kth eigenvalue is the sign of $1 + \epsilon_k q^a$. If $|q| < 1$, every such numerator is positive. At $q = \pm 1$, exactly r numerators are nonzero and positive. If $|q| > 1$, exactly r of the numerators are positive and the remaining $d - r$ are negative, with the parity interchanged when necessary as q crosses from the positive to the negative half-line. The stated inertia follows from the orthogonal diagonalization in Theorem 3.2. □

Theorem 8.2 (exact real spectral conditioning)**.** *For a nonsingular real matrix M, write $\kappa_2(M) = \|M\|_2\|M^{-1}\|_2$. Let $a \geq 3$, let $q \in \mathbb{R} \setminus \{\pm 1\}$, and put*

$$s = \begin{cases} |q|, & |q| \leq 1, \\ |q|^{-1}, & |q| > 1, \end{cases} \qquad c_j = \cos\frac{j\pi}{a}.$$

Then $0 \leq s < 1$ and $\kappa_2(C_a(q)) = \kappa_2(C_a(s))$. If a is odd, then

$$\kappa_2(C_a(q)) = \frac{1+s^a}{1-s^a}\frac{1+2sc_1+s^2}{1-2sc_1+s^2}. \tag{33}$$

If $a \geq 4$ is even, let $\rho_a \in (0,1)$ be the unique solution of

$$c_1 - c_2 = \rho_a^{a-1}\big(1 + \rho_a(c_1+c_2) + \rho_a^2\big). \tag{34}$$

Then

$$\kappa_2(C_a(q)) = \begin{cases} \dfrac{1+2sc_1+s^2}{1-2sc_1+s^2}, & 0 \leq s \leq \rho_a, \\ \dfrac{1+s^a}{1-s^a}\dfrac{1+2sc_2+s^2}{1-2sc_1+s^2}, & \rho_a \leq s < 1. \end{cases} \tag{35}$$

For $a = 2$, one has $C_2(q) = (1)$ and $\kappa_2(C_2(q)) = 1$.

After reduction to s, the largest eigenvalue is always the $k = 1$ sine mode. For odd a, the smallest is the $k = a-1$ mode. For even a, the smallest is the $k = a-1$ mode for $s \leq \rho_a$ and the $k = a-2$ mode for $s \geq \rho_a$. In particular, as $s \uparrow 1$,

$$\kappa_2(C_a(q)) \sim \begin{cases} \dfrac{2(1+c_1)}{a(1-c_1)}\dfrac{1}{1-s}, & a \text{ odd}, \\ \dfrac{2(1+c_2)}{a(1-c_1)}\dfrac{1}{1-s}, & a \text{ even}. \end{cases} \tag{36}$$

Proof. The sign conjugation (10) gives $\kappa_2(C_a(q)) = \kappa_2(C_a(|q|))$. For $t > 1$, the reciprocal law (12) reads

$$C_a(t) = t^{a-2} J_a C_a(t^{-1}).$$

Since J_a is orthogonal and scalar multiplication does not change a condition number, this reduces every real nonsingular parameter to $0 \leq s < 1$. By Theorem 8.1, $C_a(s)$ is positive definite, so its spectral condition number is the ratio of its largest and smallest sine-basis eigenvalues.

Put

$$d_k(s) = 1 - 2s\cos\theta_k + s^2.$$

The sequence $d_k(s)$ is increasing in k. Moreover $1 + \epsilon_k s^a \le 1 + s^a$, and therefore

$$\lambda_{a,k}(s) \le \frac{1+s^a}{d_k(s)} \le \frac{1+s^a}{d_1(s)} = \lambda_{a,1}(s).$$

Thus $k = 1$ is always maximal.

Suppose first that a is odd. Then $a-1$ is even, so the $k = a-1$ mode has the smaller numerator $1 - s^a$ and the largest denominator. It is therefore minimal. Dividing $\lambda_{a,1}(s)$ by $\lambda_{a,a-1}(s)$ proves (33).

Now let a be even. Among the odd modes the minimum occurs at $k = a-1$, and among the even modes it occurs at $k = a-2$. Hence only

$$\lambda_{a,a-1}(s) = \frac{1+s^a}{1+2sc_1+s^2}, \qquad \lambda_{a,a-2}(s) = \frac{1-s^a}{1+2sc_2+s^2}$$

need be compared. Equality is equivalent, for $s > 0$, to (34). The right-hand side of that equation is strictly increasing on $[0,1]$, begins at 0, and at 1 equals $2 + c_1 + c_2 > c_1 - c_2$. Thus the crossover ρ_a exists and is unique. The $k = a-1$ mode is smaller below it, while the $k = a-2$ mode is smaller above it, which gives (35). Finally, (36) follows from $1 - s^a \sim a(1-s)$ and evaluation of the remaining factors at $s = 1$. □

9. The singular specialization $q = 1$

At $q = 1$,

$$C_a(1)_{\ell m} = \min(\ell, m, a-\ell, a-m).$$

The anti-mirror vectors span the kernel. Folding to the mirror-invariant subspace gives an ordinary discrete Green matrix.

Theorem 9.1 (spectrum and pseudo-determinant)**.** *Put $r = \lfloor a/2 \rfloor$. The nonzero eigenvalues of $C_a(1)$ are*

$$\lambda_k = \frac{1}{1 - \cos((2k-1)\pi/a)}, \qquad 1 \le k \le r.$$

The remaining $\lceil a/2 \rceil - 1$ eigenvalues are zero. If pdet *denotes the product of the nonzero eigenvalues, then*

$$\operatorname{pdet} C_a(1) = 2^{\lfloor (a-1)/2 \rfloor}.$$

Proof. At $q = 1$, (5) vanishes for even k. For $k = 2j - 1$ it gives

$$\lambda_{a,2j-1}(1) = \frac{2}{2 - 2\cos((2j-1)\pi/a)} = \frac{1}{1 - \cos((2j-1)\pi/a)}.$$

There are r such odd indices. The product of these nonzero eigenvalues is the determinant of the mirror-symmetric restriction, already computed in the proof of Theorem 5.2: it is 2^r for $a = 2r + 1$ and 2^{r-1} for $a = 2r$. This is $2^{\lfloor (a-1)/2 \rfloor}$. □

Corollary 9.2 (Moore–Penrose inverse)**.** *Let $o(\ell) = \min(\ell, a - \ell)$ and let $G_a = C_a(1)^+$. In operator form,*

$$G_a = \frac{1}{2} K_a(1) P_+ = \frac{1}{4} K_a(1)(I + J_a). \tag{37}$$

Equivalently, if $a = 2r + 1$, then

$$(G_a)_{\ell m} = \begin{cases} 1/2, & o(\ell) = o(m) < r, \\ 1/4, & o(\ell) = o(m) = r, \\ -1/4, & |o(\ell) - o(m)| = 1, \\ 0, & \textit{otherwise.} \end{cases}$$

If $a = 2r$, then

$$(G_a)_{\ell m} = \begin{cases} 1, & o(\ell) = o(m) = r, \\ -1/2, & \{o(\ell), o(m)\} = \{r - 1, r\}, \\ 1/2, & o(\ell) = o(m) < r, \\ -1/4, & |o(\ell) - o(m)| = 1 \textit{ and } \max(o(\ell), o(m)) < r, \\ 0, & \textit{otherwise.} \end{cases}$$

Moreover $C_a(1)G_a = G_a C_a(1)$ is the orthogonal projection onto the mirror-invariant subspace.

Proof. At $\zeta = 1$ one has $\sigma = -1$, so (29) is exactly (37). Expanding this sparse operator entrywise gives the two displayed formulas. □

Corollary 9.3 (the second real singular point)**.** *One has*

$$C_a(-1) = D_a C_a(1) D_a, \qquad C_a(-1)^+ = D_a C_a(1)^+ D_a.$$

In particular the nonzero spectrum and pseudo-determinant at $q = -1$ agree with those at $q = 1$, and the sparse formula of Corollary 9.2 transfers by this sign conjugation.

Proof. Specialize the global conjugation law (10) at $q = 1$ and use the invariance of the Moore–Penrose inverse under orthogonal conjugation. □

10. Relation to prior work and reproducibility

10.1. Relation to prior work

The starting identity (2), and hence the basic Dynkin Hilbert-series formula, belongs to established preprojective-algebra theory [8, 9]; related non-Dynkin Hilbert-series results appear in [5]. Etingof and Eu used the graded structure of ADE preprojective algebras in their computation of Hochschild and cyclic homology [4]. Anundsen and Sandøy recently generalized the associated graded-Euler-characteristic framework to higher preprojective algebras, explicitly using graded Cartan matrices as an input [1]. Thus neither the existence of the graded Cartan matrix nor its determinant-level Hilbert-series role is claimed as new here.

Chebyshev polynomials on Dynkin adjacency matrices were studied by Erdmann and Schroll [3], and the fusion terminology used for the coefficient basis is standard [11]. For general context on Smith normal form as a refinement of determinant and rank data, see Stanley's survey [10]. The nearest matrix-theoretic comparisons are q-Cartan normal forms for other classes of algebras [2] and deformed Cartan matrices for generalized preprojective algebras [6].

The contribution claimed here is the exact matrix analysis of the type-A polynomial specialization: the invariant-factor chains in Theorem 7.2, their determinantal-divisor, Fitting-ideal, and reduced cyclotomic-cokernel consequences; the complete cyclotomic rank and coordinate kernel stratification; and the explicit generic and singular Green operators, pseudodeterminants, real inertia, and condition numbers. The author is not aware of a prior statement of these invariant-factor chains or of this full singular matrix package. That priority statement is deliberately limited to the public literature checked for this release and remains subject to specialist database and referee review.

10.2. Computational verification

The primary verification script constructs $C_a(q)$ from (1) and checks the Dynkin factorization, determinant and Smith forms over $\mathbb{Q}[q]$, exact ranks over cyclotomic fields, residues, real inertias and condition numbers, the $q = 1$ spectrum and Moore–Penrose identities, and the cyclotomic Moore–Penrose formula. Independent audits reconstruct the graded matrix from

stable module maps and from doubled-quiver paths modulo the preprojective relations; verify termination and uniqueness of the declared normal-form rewrite; check Frobenius reciprocity, the Chebyshev centralizer, and sign conjugation; and test the cyclotomic pseudoinverse, pseudodeterminants, conditioning formulas, and worked Smith examples. All routes agree in the declared finite ranges. These computations are verification aids; every theorem is proved in the manuscript and no proof depends on a finite dimension cutoff.

OpenAI ChatGPT and Codex, accessed through their web services during the preparation of the work, assisted with exploratory symbolic calculations and the development of verification code. No model output is treated as mathematical evidence: the included programs are inspectable and deterministic, the recorded computations were rerun in the tested environment, and all citations were checked against the cited source records. The programs, small-case table, and tested-environment record are collected as *Online Resource 1*.

CRediT authorship contribution statement

Aaron Nagy: Conceptualization, Formal analysis, Investigation, Methodology, Validation, Writing – original draft, Writing – review and editing.

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Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data and code availability

No external data were used. Online Resource 1 contains the main exact verification program; independent stable-module and direct quiver-path audits of the graded entry formula; exhaustive normal-form rewrite, Frobenius-reciprocity, Chebyshev-centralizer, cyclotomic-pseudoinverse, cyclotomic-pseudodeterminant, real-conditioning, and worked-example audits;

a deterministic small-case generator and table; and the tested-environment record. The programs check the polynomial matrix identities, Smith forms, cyclotomic ranks and kernels, residues, algebraic pseudodeterminants, inertia, Moore–Penrose formulas, coefficient reversal, and the declared rewrite mechanism. The proofs in the manuscript do not depend on a finite computational cutoff.

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work, the author used OpenAI ChatGPT and Codex to support symbolic exploration, verification-code development, literature-query formulation, manuscript organization, and language editing. After using these tools, the author reviewed and edited the material as needed, verified the references against source records, checked every mathematical claim against the proofs and exact computations supplied with the manuscript, and takes full responsibility for the content of the published article.

References

[1] J. W. Anundsen and M. H. Sandøy, Hochschild (co)homology and cyclic homology via a graded Euler characteristic with applications to higher preprojective algebras, arXiv preprint arXiv:2606.26255 (2026), doi:10.48550/arXiv.2606.26255.

[2] C. Bessenrodt and T. Holm, q-Cartan matrices and combinatorial invariants of derived categories for skewed-gentle algebras, *Pacific J. Math.* **229** (2007), no. 1, 25–47, doi:10.2140/pjm.2007.229.25.

[3] K. Erdmann and S. Schroll, Chebyshev polynomials on symmetric matrices, *Linear Algebra Appl.* **434** (2011), no. 12, 2475–2496, doi:10.1016/j.laa.2010.12.034.

[4] P. Etingof and C.-H. Eu, Hochschild and cyclic homology of preprojective algebras of ADE quivers, *Mosc. Math. J.* **7** (2007), no. 4, 601–612, doi:10.17323/1609-4514-2007-7-4-601-612.

[5] P. Etingof and C.-H. Eu, Koszulity and the Hilbert series of preprojective algebras, *Math. Res. Lett.* **14** (2007), no. 4, 589–596, doi:10.4310/MRL.2007.v14.n4.a4.

[6] R. Fujita and K. Murakami, Deformed Cartan matrices and generalized preprojective algebras I: finite type, *Int. Math. Res. Not. IMRN* (2023), no. 8, 6924–6975, doi:10.1093/imrn/rnac054.

[7] O. Iyama and X. Zhang, Classifying τ-tilting modules over the Auslander algebra of $K[x]/(x^n)$, *J. Math. Soc. Japan* **72** (2020), no. 3, 731–764, doi:10.2969/jmsj/75117511.

[8] A. Malkin, V. Ostrik, and M. Vybornov, Quiver varieties and Lusztig's algebra, *Adv. Math.* **203** (2006), no. 2, 514–536, doi:10.1016/j.aim.2005.05.002.

[9] H. Minamoto, The Hilbert series of the preprojective algebras, *J. Algebra* **691** (2026), 730–740, doi:10.1016/j.jalgebra.2025.12.002.

[10] R. P. Stanley, Smith normal form in combinatorics, *J. Combin. Theory Ser. A* **144** (2016), 476–495, doi:10.1016/j.jcta.2016.06.013.

[11] E. Verlinde, Fusion rules and modular transformations in two-dimensional conformal field theory, *Nuclear Phys. B* **300** (1988), 360–376, doi:10.1016/0550-3213(88)90603-7.